Walking with Fabrice among large deviations and sum rules

Alain Rouault (UVSQ-Paris-Saclay)

2023 April 5th

Le Croisic - MASCOT 2023

Plan

Introduction

Fabrice is a statistician
: How to estimate a (probability) measure ?

1) Knowing independent observations → sampling i.i.d.

2) Knowing stationary observations → periodogram.

3) Knowing some moments
Let $E = [0, 1]$ or $\mathbb{T}$ (the unit circle). Let μ be a probability measure on E. Its k-th moment is

$$m_k(\mu) = \int_E x^k d\mu(x) .$$

Data : $m_1(\mu), \ldots, m_n(\mu)$

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Common threads :

- discrete random measures

$$\mu^{(n)} = \frac{1}{n}\sum_{1}^{n} Z_i^{(n)}\delta_{X_i^{(n)}} \text{ with } \mathbb{E}X_i^{(n)} = 1\,.$$

- large deviations

$$\mathbb{P}(\mu^{(n)} \cong \mu) \cong \exp -a_n I(\mu) \;\; (a_n \uparrow \infty)$$

- role played by the Kullback-Leibler information

$$K(P|Q) = \begin{cases} \int \log \frac{dP}{dQ} dP & \text{if } P \ll Q \text{ and } \log \frac{dP}{dQ} \in L^1(P), \\ \infty \;\text{ otherwise,} \end{cases} \quad (1)$$

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1) Maximum Entropy for the Mean
   + D. Dacunha-Castelle, + E. Gassiat (Orsay)
2) Random moment problems
   (+ L-V. Lozada-Chang (La Habana), + H. Dette (Bochum))
3) Empirical spectral measures of stationary Gaussian processes
   + B. Bercu (Bordeaux), + M. Lavielle (INRIA), + AR, + M. Zani (Orléans)
4) Spectral measures, Random matrices and Sum rules
   + J. Nagel (Dortmund), + AR

Plan

MEM

Estimate μ from noisy observations of some moments.

- A reference probability P
- Build a sequence
$$\frac{1}{n}\sum_{1}^{n}\delta_{x_k^{(n)}} \Rightarrow_{n\to\infty} P\,.$$
- Choose a law F and Z_k i.i.d. with law F and consider
$$\nu_n = \frac{1}{n}\sum_{1}^{n} Z_k\delta_{x_k}$$
- Compute
$$P_n^{MEM} = \text{Argmin}\{\mathcal{K}(R; F^{\otimes n}\}$$
among the laws R such that the mean moments of ν_n under R satisfy the constraint.
- Define the estimator $\hat{\nu}_n^{MEM} = \mathbb{E}_{P_n^{MEM}}\nu_n$.

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Random moment problem

Fabrice is a probabilist

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$$\mathbf{m}_k(\mu) = (m_1(\mu), \ldots, m_k(\mu)) .$$

The sets of moments are

$$\mathbb{M}_k = \{\mathbf{m}_k(\mu); \mu \in \mathcal{M}_1(E)\} \quad (k \geq 1)$$

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$$m_{k+1}^{+} = \sup\{m_{k+1}(\mu); \mu \text{ s.t.} \mathbf{m}_k(\mu) = (m_1, \ldots, m_k)\}$$
$$m_{k+1}^{-} = \inf\{m_{k+1}(\mu); \mu \text{ s.t. } \mathbf{m}_k(\mu) = (m_1, \ldots, m_k)\}$$

There exists two discrete measures $\mu_k^{\pm}$ corresponding to

$$(m_1, \ldots, m_k, m_k^{\pm}) .$$

We can equip $\mathbb{M}_k$ with the uniform measure and consider the above (random) measures.

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Let $\mathcal{H}$ be an Hilbert space, A a bounded self-adjoint operator and e a cyclic unitary vector. From the spectral theorem, there exists a unique p.m. μ compactly supported such that

$$\langle e, A^k e\rangle_{\mathcal{H}} = \int_{\mathbb{R}} x^k d\mu(x)\ ,\ k \geqslant 0\,.$$

If we define the equivalence relation

$$(\mathcal{H}, A, e) \sim (\mathcal{K}, B, f)$$

iff there exists an isometry $V : \mathcal{H} \to \mathcal{K}$ such that $B = VAV^{-1}$ and $f = Ve$ then μ is an invariant of the class.

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2 remarkable elements in each class :

- $(L^2(d\mu),\ h \mapsto (x \mapsto xh(x))\ , 1)$
- (J, e_1) where J is a tridiagonal matrix and $e_1 = (1, 0, 0, \ldots)$.

Finite dimension

Let A_n be a $n \times n$ self-adjoint matrix, with eigenvalues $(\lambda_j)_{j=1}^n$ and (unitary) eigenvectors $(\psi_j)_{j=1}^n$. Assume that $e = e_1$ is cyclic. Then the sequence

$$\left((A_n)^k\right)_{11}\ \ k \geqslant 1$$

is the sequence of moments of

$$\mu_w^{(n)} = \sum_{j=1}^n w_j \delta_{\lambda_j}\ ,\ \ w_j := |\langle \psi_j, e_1\rangle|^2\ . \qquad (2)$$

We can encode $\mu_w^{(n)}$ by two n-uplets $(w_1, \ldots, w_n)$ (weights) and $(\lambda_1, \ldots, \lambda_n)$.

There is another encoding based on orthogonal polynomials.

In the basis $(\varepsilon_1, \ldots, \varepsilon_n)$ obtained by the Gram-Schmidt procedure applied to $(e, A_n e, \ldots, A_n^{n-1} e)$, the matrix A_N becomes

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In the basis $(\epsilon_1, \dots, \epsilon_n)$ obtained by the Gram-Schmidt procedure applied to $(e, A_n e, \dots, A_n^{n-1} e)$, the matrix A_N becomes

$$J_n = \begin{pmatrix} b_1 & a_1 & 0 & \dots & 0 \\ a_1 & b_2 & a_2 & \dots & 0 \\ 0 & a_2 & b_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & a_{n-1} & b_n \end{pmatrix}$$

with $a_j > 0$ for all j.
If

$$(e, Ae, \dots, A^{n-1}e) \longleftrightarrow (1, x, \dots, x^{n-1})$$

then, by Gram-Schmidt

$$(\epsilon_1, \dots, \epsilon_n) \longleftrightarrow \big(1, p_1(x), \dots, p_{n-1}(x)\big)$$

These polynomials satisfy the recursion :

$$xp_j(x) = a_{j+1}p_{j+1}(x) + b_{j+1}p_j(x) + a_jp_{j-1}(x)\,.$$

Summarizing, we have two remakable random measures :

$$\mu_u^{(n)} = \frac{1}{n}\sum_1^n \delta_{\lambda_j}$$

(empirical spectral distribution), whose moments are

$$m_k(\mu_u^{(n)}) = \frac{1}{n}\mathrm{tr}(A_n)^k$$

and

$$\mu_w^{(n)} = \sum_1^n w_j \delta_{\lambda_i}$$

(spectral measure of (A_n, e_1)), whose moments are

$$m_k(\mu_w^{(n)}) = \left((A_n)^k\right)_{11}$$

Randomization

- Suppose the distribution of M_n has the GUE-density

$$\mathcal{Z}_n^{-1} \exp -\frac{n}{2} \mathrm{tr}\,(M^2)$$

- Dumitriu-Edelman ('02) proved that the Jacobi parameters are independent and

$$b_k^{(n)} \sim \mathcal{N}(0; n^{-1}) \quad (1 \leqslant k \leqslant n),$$
$$(a_k^{(n)})^2 \sim \text{Gamma}\,(n-k; n^{-1}) \;\; (1 \leqslant k \leqslant n-1)\,.$$

Note that $b_k^{(n)} \to 0$, $a_k^{(n)} \to 1$, the Jacobi coefficients of SC :

$$SC(dx) = \frac{1}{2\pi}\sqrt{(4-x^2)_+}\,dx\,.$$

Theorem (GR '11)

$\mu_{\mathtt{w}}^{(n)}$ *satisfies the LDP with speed* n *with rate function*

$$\mathcal{I}_{\text{coeff}} = \sum_{1}^{\infty} \frac{1}{2} b_k^2 + \sum_{1}^{\infty} G(a_k^2)\ ,\ \ G(x) = x - 1 - \log x\,.$$

LDP for the "measure side", general potential (no gap)

- M_n random complex Hermitian $n \times n$ matrix with density

$$(\mathcal{Z}_n^V)^{-1} \exp(-n \mathrm{tr}\, V(M))$$

- Potential $V : \mathbb{R} \to (-\infty, \infty]$ smooth, e.g. $V(x) = x^2/2$, (GUE).
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$$\mu_w^{(n)} = \sum_1^n w_i \delta_{\lambda_i}$$

with $w_i = |U_{1,i}|^2$ for U unitary matrix of eigenvectors.
- The joint density of eigenvalues is

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and $(w_1, \dots, w_n)$ is uniformly distributed on the simplex, and independent of the eigenvalues.

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Question : how to get LDP for

$$\mu_{\mathtt{w}}^{(n)} = \sum_{k=1}^{n} \mathtt{w}_i \delta_{\lambda_i}$$

directly ?

First point : Asymptotics of the ESD

$$\mu_u^{(n)} := \frac{1}{n}\sum_1^n \delta_{\lambda_j}$$

satisfies :

- $\lim_N \mu_u^{(n)} = \mu_V$ in probability with μ_V compactly supported by $[a_V, b_V]$ (equilibrium measure a.k.a. density of states)
- Ben Arous, Guionnet ('97) : $(\mu_u^{(n)})$ satisfies the LDP with speed n^2 and rate function

$$\mathcal{I}^{ESD}(\mu) = \int V(x)d\mu(x) - \int\int \log|x-y|d\mu(x)d\mu(y) - c_V$$

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Second point : decoupling

At scale n, the measure $\mu_u^{(n)}$ is quasi-deterministic, and the randomness comes essentially from the weights w_k's.

The weights w_k are not independent but

$$(w_1, \ldots, w_n) \stackrel{(d)}{=} \left(\frac{\gamma_1}{\gamma_1 + \cdots + \gamma_n}, \ldots, \frac{\gamma_n}{\gamma_1 + \cdots + \gamma_n} \right)$$

where the γ_k's are independent, $\exp(1)$. So, we can write

$$\mu_w^{(n)} \stackrel{(d)}{=} \frac{\tilde{\mu}_n}{\int d\tilde{\mu}_n} \text{ with } \tilde{\mu}_n = \sum_{k=1}^{n} \gamma_k \delta_{\lambda_k},$$

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$\mu \sim \{\int f d\mu ; f \in \mathcal{C}_b\}$ (Gắrtner-Ellis approach)

$$\mathbb{E}\left[\exp\left(n\int f d\tilde{\mu}_n\right)\right] = \mathbb{E}\left[\prod_{k=1}^{n}\exp(\gamma_k f(\lambda_k))\right]$$

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with $L(x) = -\log(1-x)$. Then, it could be expected that

$$\frac{1}{n}\log\mathbb{E}\left[\exp\left(n\int f d\tilde{\mu}_n\right)\right] \to \int (L\circ f) d\mu_V$$

and then the LDP for $\tilde{\mu}_n$ would be

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But there are contributions of the outliers, due to their LDP with speed n with rate function $\mathcal{F}_V^{\pm}$. This function is the effective potential (up to a constant). When the potential is quadratic, it is

$$\mathcal{F}_H^{\pm}(x) = \int_2^{|x|} \sqrt{t^2 - 4}dt\,,$$

i.e. if $\lambda_1^{(n)} > \lambda_2^{(n)} > \cdots > \lambda_r^{(n)}$, then

$$\mathbb{P}\left(\lambda_1^{(n)} \simeq E_1^+, \ldots, \lambda_r^{(n)} \simeq E_r^+\right) = \exp -n\left[\sum_1^r \mathcal{F}_H^+(E_i^+) + o(1)\right]$$

Theorem (GNR '16)

Under assumptions on V, $(\mu_{\mathtt{w}}^{(n)})$ *satisfies the LDP in the scale* N *with good rate function*

$$\mathcal{I}_{\mathsf{meas}}(\mu) = \mathcal{K}(\mu_V \,|\mu) + \sum_k \mathcal{F}_V(E_k^+) + \sum_k \mathcal{F}_V(E_k^-)$$

for probability measures μ *on* $\mathbb{R}$ *satisfying*

$$\mathit{Supp}(\Sigma) = [a_V, b_V] \cup \{E_k^-\}_{k=1}^{K^-} \cup \{E_k^+\}_{j=1}^{K^+}$$

where K^+ *(resp.* K^-*) is* 0*, finite or infinite and* $E_k^- \uparrow a_V$ *and* $E_k^+ \downarrow b_V$ *are isolated points of the support,* μ_V *is the equilibrium measure and* $[a_V, b_V]$ *is its support.*

First sum rule

We have 2 LDPs for the same object under two different encodings I_{coeff} via the coefficients encoding and I_{meas} via the weight + support encoding. By uniqueness we obtain $I_{coeff}(\mu) = I_{meas}(\mu)$. We recover the

Killip-Simon sum rule (2003)

$$\mathcal{K}(SC|\,\mu) + \sum_k \mathcal{F}(E_k^+) + \sum_k \mathcal{F}(E_k^-) = \sum_j G(a_j^2) + \frac{b_j^2}{2},$$

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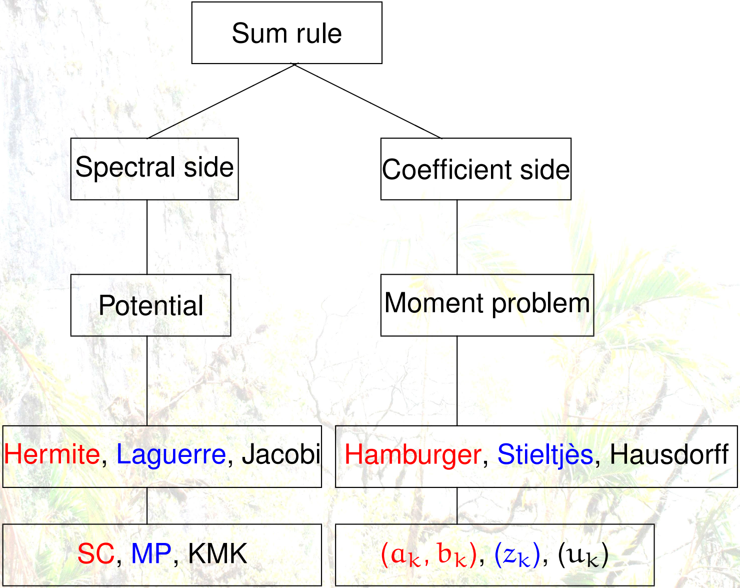
Sum rule
Spectral side
Coefficient side
Potential
Moment problem
Hermite, Laguerre, Jacobi
Hamburger, Stieltjès, Hausdorff
SC, MP, KMK
(a_k, b_k), (z_k), (u_k)

On the unit circle $\mathbb{T}$ we have three sum rules :

Recovering

Szegő-Verblunsky formula (1915-1936)

$$\mathcal{K}(\mathrm{UNIF} \mid \mu) = \sum_j -\log(1 - |\alpha_j|^2)$$

New sum rules

Gross-Witten, with potential $\mathcal{V}(e^{i\theta}) = g \cos\theta$.

Hua-Pickrell, with potential $\mathcal{V}(e^{i\theta}) = d \log|1 - e^{i\theta}|^2$.

Conclusion : Fabrice is also a spectral analyst !

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Merci pour votre attention!

Thanks for your attention!

and


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