Walking with Fabrice among large deviations and sum rules

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Le Croisic - MASCOT 2023



Plan

1 Introduction

Fabrice is a statistician

- How to estimate a (probability) measure?
- 1) Knowing independent observations \rightarrow sampling i.i.d.
- 2) Knowing stationary observations \rightarrow periodogram.
- 3) Knowing some moments
- Let E = [0,1] or T (the unit circle). Let μ be a probability measure on E. Its k-th moment is

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 $m_{k}(u) \Rightarrow (x \neq du(x)).$

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Let E = [0, 1] or \mathbb{T} (the unit circle). Let μ be a probability measure on E. Its k-th moment is

$$\mathfrak{m}_{k}(\mu) = \int_{E} x^{k} d\mu(x) \, .$$

Data : $m_1(\mu), ..., m_n(\mu)$.

Common threads :

discrete random measures

 $\mu(\mathbf{u}) = \frac{1}{n} \sum_{i=1}^{n} Z_i^{(n)} \delta_{\chi_i^{(n)}}$ with $\mathbb{E} X_i^{(n)} = 1$.

large deviations

 $\mathbb{P}(\mu^{(n)} \cong \mu) \cong \exp -a_n I(\mu) \ (a_n \uparrow \infty)$

role played by the Kullback Leibler information

 $\int \log \frac{dP}{dQ} dP$ if $P \ll Q$ and $\log \frac{P}{dQ}$ otherwise,

Common threads :

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large deviations

role played by the Kullback Leibler information

 $\int \log \frac{dP}{dQ} dP \quad (i, P \neq Q \text{ and } \log R)$

Common threads :

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$$\mathbb{P}(\mu^{(n)} \cong \mu) \cong \exp -a_n I(\mu) \ (a_n \uparrow \infty)$$

role played by the Kullback Leibler information $P(Q) = \begin{cases} \int \log \frac{dP}{dQ} dP & \text{if } P \neq Q \text{ and } \log P \\ \infty & \text{otherwise,} \end{cases}$

Common threads :

- discrete random measures

$$\mu^{(n)} = \frac{1}{n} \sum_{1}^{n} Z_{i}^{(n)} \delta_{X_{i}^{(n)}} \text{ with } \mathbb{E} X_{i}^{(n)} = 1 \,.$$

- large deviations

$$\mathbb{P}(\mu^{(n)} \cong \mu) \cong \exp -a_n \mathbf{I}(\mu) \ (a_n \uparrow \infty)$$

- role played by the Kullback-Leibler information

$$\mathcal{K}(P|Q) = \begin{cases} \int \log \frac{dP}{dQ} dP & \text{if } P \ll Q \text{ and } \log \frac{dP}{dQ} \in L^1(P), \\ \infty \text{ otherwise,} \end{cases}$$

- 1) Maximum Entropy for the Mean
 - + D. Dacunha-Castelle, + E. Gassiat (Orsay)
- Random moment problems
 (+ L-V. Lozada-Chang (La Habana), + H. Dette (Bochum))
- 3) Empirical spectral measures of stationary Gaussian processes
 + B. Bercu (Bordeaux), + M. Lavielle (INRIA), + AR, + M. Zani (Orléans)
- 4) Spectral measures, Random matrices and Sum rules
 - + J. Nagel (Dortmund), + AR



MEM

Estimate μ from noisy observations of some moments.

MEM

- A reference probability P
- Build a sequence

$$\frac{1}{n}\sum_{1}^{n}\delta_{\mathbf{x}_{k}^{(n)}}\Rightarrow_{n\to\infty}\mathsf{P}.$$

Choose a law F and Z_k i.i.d. with law F and consider

$$\mathbf{v}_n = \frac{1}{n} \sum_{1}^{n} \mathsf{Z}_k \delta_{\mathbf{x}_k}$$

Compute

$$\mathsf{P}_{\mathsf{n}}^{\mathsf{MEM}} = \mathsf{Argmin}\{\mathcal{K}(\mathsf{R};\mathsf{F}^{\otimes \mathsf{n}}\}\)$$

among the laws R such that the mean moments of ν_n under R satisfy the constraint.

• Define the estimator $\hat{v}_n^{MEM} = \mathbb{E}_{P_n^{MEM}} v_n$.

Plan

3 Random moment problem

Random moment problem

Fabrice is a probabilist

Let $E \leftarrow [0, 1]$ or \mathbb{T} (the unit circle). Let μ be a probability measure on E. Its K-th moment is

 $\mathbf{m}_{\mathbf{k}}(\mathbf{\mu}) = \int_{\mathbf{F}} \mathbf{x}^{\mathbf{k}} d\mathbf{\mu}(\mathbf{x}) \, d$

The vector of the first \mathbf{k} moments is denoted by

 $\mathbf{m}_{k}(\mu) = (\mathfrak{m}_{1}(\mu), \dots, \mathfrak{m}_{k}(\mu))$

The sets of moments are

 $\mathbf{1}_{\mathbf{k}} = \{\mathbf{m}_{\mathbf{k}}(\boldsymbol{\mu}), \boldsymbol{\mu} \in \mathcal{M}_{\mathbf{k}}(\mathbf{E})\} \quad (\mathbf{k} \ge 1)$

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Random moment problem

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The vector of the first k moments is denoted by

$$\mathbf{m}_{k}(\boldsymbol{\mu}) = (\mathfrak{m}_{1}(\boldsymbol{\mu}), \dots, \mathfrak{m}_{k}(\boldsymbol{\mu}))$$

The sets of moments are

 $\mathbb{M}_{k} = \{ \mathbf{m}_{k}(\mu); \mu \in \mathcal{M}_{1}(E) \} \quad (k \ge 1) .$

They are compact convex sets.

We can equip M), with the uniform measure and consider the above -

If $\mathsf{E}=[0,1]\ \mathbb{M}_k$ and if $(m_1,\ldots,m_k)\in$ Int \mathbb{M}_k then we can define

$$\begin{split} \mathbf{m}_{k+1}^+ &= \sup\{\mathbf{m}_{k+1}(\mu); \mu \text{ s.t.} \mathbf{m}_k(\mu) = (\mathbf{m}_1, \dots, \mathbf{m}_k)\}\\ \mathbf{m}_{k+1}^- &= \inf\{\mathbf{m}_{k+1}(\mu); \mu \text{ s.t. } \mathbf{m}_k(\mu) = (\mathbf{m}_1, \dots, \mathbf{m}_k)\} \end{split}$$

There exists two discrete measures μ_k^{\pm} corresponding to

 $(\mathfrak{m}_1,\ldots,\mathfrak{m}_k,\mathfrak{m}_k^{\pm})$.

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$$m_{k+1}^{+} = \sup\{m_{k+1}(\mu); \mu \text{ s.t.} \mathbf{m}_{k}(\mu) = (m_{1}, \dots, m_{k})\}$$

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4 Spectral measures

Let 9t be an Hilbert space, A a bounded self-adjoint operator and e a cyclic unitary vector. From the spectral theorem, there exists a unique orn, μ compactly supported such that

$${}^{\mathbf{k}}\mathbf{q}
angle_{\mathcal{H}}=\int_{\mathbb{R}}\mathbf{x}^{\mathbf{k}}d\mu(\mathbf{x})$$
 , $\mathbf{k}\geqslant0$.

If we define the equivalence relation

 $(\mathcal{H}, \mathcal{A}, \mathbf{e}) = (\mathcal{K}, \mathcal{B}, \mathbf{f})$

iff there exists an isometry $V: \mathcal{H} \to \mathcal{H}$ such that $B = V A V^{-1}$ and $f = Ve^{-1}$, then use an invariant of the class.

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$$\langle e, A^k e
angle_{\mathcal{H}} = \int_{\mathbb{R}} x^k d\mu(x) , \ k \geqslant 0.$$

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 $(\mathcal{K}, \mathbf{A}, \mathbf{e}) \approx (\mathcal{K}, \mathbf{B}, \mathbf{f})$

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$$(\mathcal{H}, \mathbf{A}, \mathbf{e}) \sim (\mathcal{K}, \mathbf{B}, \mathbf{f})$$

iff there exists an isometry $V : \mathcal{H} \to \mathcal{K}$ such that $B = VAV^{-1}$ and f = Ve then μ is an invariant of the class.

- $(L^2(a\mu), h \mapsto (x \mapsto xh(x)), 1)$
- e_1 . We have this a tridiagonal matrix and $e_1=(1,0,0,\dots)$
- Let A_n be a $n \times n$ self-adjoint matrix, with eigenvalues $(\lambda_j)_{j=1}^n$ and (unitary) eigenvectors $(\psi_j)_{j=1}^n$. Assume that $e = e_1$ is cyclic. Then the sequence
 - $\left((A_n)^k \right)_{11} k \ge 1$
- is the sequence of moments of
 - $\mathbb{I}_{w}^{(n)} := \sum \mathbb{W}_{j} \delta_{\lambda_{j}} \langle \mathbb{W}_{j} := |\langle \psi_{j}, e_{1} \rangle|^{2} \, .$
 - . We can encode μ_{1}^{m} , by two n_{1} uplets ($w_{1},\ldots,w_{n})$ (weights) and
- There is another encoding based on orthogonal polynomials. In the basis (e_n) , (e_n) , obtained by the Gram-Schmidt procedure applied to $(e_n A_n e_1)$, $A_n = 1e_1$, the matrix A_{N_n} becomes

- $\blacktriangleright (L^2(d\mu), \ h \mapsto (x \mapsto xh(x)) \ , 1)$
- ▶ (ℓ^2, J, e_1) where J is a tridiagonal matrix and $e_1 = (1, 0, 0, ...)$.

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There is another encoding based on orthogonal polynomials. In the basis (e_1 , e_n), obtained by the Gram-Schmidt procedure applied to (e_1 , e_n , e_n), the matrix A $_N$ becomes

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We can encode $\mu_w^{(n)}$ by two n-uplets (w_1, \ldots, w_n) (weights) and $(\lambda_1, \ldots, \lambda_n)$.

There is another encoding based on orthogonal polynomials. In the basis (e_1 , e_n) obtained by the Gram-Schmidt procedure applied to (e. A, e), $A_n^{-1}e$), the matrix A_n becomes

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$$\mathfrak{u}_{\mathtt{w}}^{(n)}:=\sum_{i=1}^{n}\mathtt{w}_{j}\delta_{\lambda_{j}}$$
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A. Rouault (LMV)

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$$J_{n} = \begin{pmatrix} b_{1} & a_{1} & 0 & \dots & 0 \\ a_{1} & b_{2} & a_{2} & \dots & 0 \\ 0 & a_{2} & b_{3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & a_{n-1} & b_{n} \end{pmatrix}$$

with $a_j > 0$ for all j. If

$$(e, Ae, \ldots, A^{n-1}e) \longleftrightarrow (1, x, \ldots, x^{n-1})$$

then, by Gram-Schmidt

$$(\epsilon_1,\ldots,\epsilon_n) \longleftrightarrow (1,p_1(x),\ldots,p_{n-1}(x))$$

These polynomials satisfy the recursion :

$$xp_j(x) = a_{j+1}p_{j+1}(x) + b_{j+1}p_j(x) + a_jp_{j-1}(x)$$

Summarizing, we have two remakable random measures :

$$\mathfrak{u}_{\mathfrak{u}}^{(n)} = rac{1}{n}\sum_{1}^{n}\delta_{\lambda_{\mathfrak{j}}}$$

(empirical spectral distribution), whose moments are

$$\mathfrak{m}_k(\mu_u^{(n)}) = \frac{1}{n} tr(A_n)^k$$

$$\mu_{\tt w}^{(n)} = \sum_1 {\tt w}_j \delta_{\lambda_i}$$

n

(spectral measure of (A_n, e_1)), whose moments are

$$\mathfrak{m}_{k}(\mu_{w}^{(n)}) = \left((A_{n})^{k} \right)_{1:}$$

Randomization

Suppose the distribution of M_n has the GUE-density

$$\mathfrak{Z}_n^{-1} \exp{-\frac{n}{2}} tr(M^2)$$

 Dumitriu-Edelman ('02) proved that the Jacobi parameters are independent and

$$\begin{split} b_k^{(n)} &\sim \mathcal{N}(0; n^{-1}) \quad (1 \leqslant k \leqslant n), \\ (a_k^{(n)})^2 &\sim \text{Gamma} \ (n-k; n^{-1}) \ (1 \leqslant k \leqslant n-1) \,. \end{split}$$

Note that $b_k^{(n)} \rightarrow 0$, $a_k^{(n)} \rightarrow 1$, the Jacobi coefficients of SC :

$$SC(dx) = \frac{1}{2\pi} \sqrt{(4-x^2)_+} dx.$$

Theorem (GR '11)

 $\mu_{\tt w}^{(n)}$ satisfies the LDP with speed n with rate function

$$\mathbb{J}_{\text{coeff}} = \sum_{1}^{\infty} \frac{1}{2} b_k^2 + \sum_{1}^{\infty} G(a_k^2) \text{ , } \ G(x) = x - 1 - \log x \, .$$



LDP for the "measure side", general potential (no gap)

 M_n random complex Hermitian $n \times n$ matrix with density

 $(\mathcal{Z}_n^V)^{-1} \exp(-\operatorname{ntr} V(M))$

 $\mu_{\rm w}^{(n)} = \sum_{i=1}^{n} w_i \delta_{\lambda_i}$

Potential $V : \mathbb{R} \to (-\infty, \infty]$ smooth, e.g. $V(x) = x^2/2$, (GUE)

with $t_{i} = |U_{1,i}|^2$ for U unitary matrix of eigenvectors. The joint density of eigenvalues is

$$Z_{n}^{(n)} = \prod_{i=1}^{n} (\lambda_{i} - \lambda_{i})^{2} \prod_{i=1}^{n} \exp(-nV)^{n}$$

ndependent of the eigenvalues

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with $w_i = |U_{1,i}|^2$ for U unitary matrix of eigenvectors. The joint density of eigenvalues is -1

 $(\lambda_{i} - \lambda_{i})^{-1} \prod (\lambda_{i} - \lambda_{i})^{-1} \prod \exp(-nV \Lambda c)$

and w_1, \dots, w_n) is uniformly distributed on the simplex and independent of the eigenvalues

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 $\mu_{w}^{(n)} = \sum_{w_{i}}^{n} w_{i} \delta_{\lambda_{i}}$

Potential V : $\mathbb{R} \to (-\infty, \infty]$ smooth, e.g. V(x) = $x^2/2$, (GUE).

with $w_{i} = |U_{1,i}|^2$ for U, unitary matrix of eigenvectors. The joint density of eigenvalues is $-\frac{1}{2}$

and w_1, \dots, w_n) is uniformly distributed on the simplex and independent of the eigenvalues

 $(Z_n)^{-1} \prod (\lambda_i - \lambda_i)^{\mu} \exp(-nV)$

 $({\mathbb Z}_n^V)^{-1} \exp(-{\mathsf{ntr}} \ V(M))$

Potential $V : \mathbb{R} \to (-\infty, \infty]$ smooth, e.g. $V(x) = x^2/2$, (GUE).

$$\mu_{\mathbf{w}}^{(n)} = \sum_{1}^{n} \mathbf{w}_{i} \delta_{\lambda_{i}}$$

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 $Z_n^{V_1-1} \prod (\lambda_i - \lambda_i)^2 \prod exp(-nV)$

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Potential $V : \mathbb{R} \to (-\infty, \infty]$ smooth, e.g. $V(x) = x^2/2$, (GUE).

$$\mu_{\mathtt{W}}^{(n)} = \sum_{1}^{n} \mathtt{w}_{\mathtt{i}} \delta_{\lambda_{\mathtt{i}}}$$

with $w_i = |U_{1,i}|^2$ for U unitary matrix of eigenvectors. The joint density of eigenvalues is

$$(Z_n^V)^{-1}\prod_{i< j}(\lambda_i - \lambda_j)^2\prod_i^n \exp(-nV(\lambda_i))$$

and (w_1, \ldots, w_n) is uniformly distributed on the simplex, and independent of the eigenvalues.

A. Rouault (LMV)

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Question : how to get LDP for

$$\mu_{\mathtt{w}}^{(\mathtt{n})} = \sum_{k=1}^{\mathtt{n}} \mathtt{w}_{\mathtt{i}} \delta_{\lambda_{\mathtt{i}}}$$

directly?

$$\mu_{u}^{(n)} \coloneqq \frac{1}{n} \sum_{1}^{n} \delta_{\lambda_{j}}$$

satisfies :

- $\lim_{N} \mu_a^{(n)} = \mu_V$ in probability with μ_V compactly supported by $\gamma[a_V, b_V]$ (equilibrium measure a.k.a. density of states)
- Ben Arous, Guionnet (97) : $(\mu_u^{(n)})$ satisfies the LDP with speed n^2 and rate function
 - $\mathcal{I}^{\text{ESD}}(\mu) = \mathcal{V}(x) d\mu(x) \int \log |x y| d\mu(x) d\mu(y) d\mu(x) d\mu(y) d\mu(x) d\mu(y) = 0$
 - But Arous, Dembo, Guionnet ('01) : The largest (resp) smallest) evaluations the LDP at right of b_V (resp. at left of a_V) with speed 1 and

$$\mu_u^{(n)} := \frac{1}{n} \sum_1^n \delta_{\lambda_j}$$

satisfies :

- lim_N $\mu_u^{(n)} = \mu_V$ in probability with μ_V compactly supported by $[a_V, b_V]$ (equilibrium measure a.k.a. density of states)
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 $\mathcal{J}^{\mathsf{ESD}}(\mu) = \int V(x) d\mu(x) - \int \int \log |x - y| d\mu(x) d\mu(y) - c_{\mathsf{V}}$

Ben Arous, Dembo, Guionnet (101) . The largest (resolvenallest) evaluations the LDP at right of b_V (resp. at left of a_V) with speed 1 and

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$$\mathcal{I}^{ESD}(\mu) = \left| V(x)d\mu(x) - \right| \left| \log |x - y|d\mu(x)d\mu(y) - c_{V} \right|$$

Ben Arous, Dembo, Guionnet ('01) : The largest (resp.smallest) ev satisfies the LDP at right of b_V (resp. at left of a_V) with speed n and rate \mathcal{F}_V^{\pm} .

At scale n, the measure $\mu_{u}^{(n)}$ is quasi-deterministic, and the randomness comes essentially from the weights w_k 's.

The weights w_k are not independent but

where the γ_k 's are independent, exp(1). So, we can write

study list un and then make the "contraction"

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with $I(x) = -\log(1-x)$. Then, it could be expected that

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But there are contributions of the outliers, due to their LDP with speed n with rate function \mathcal{F}_V^{\pm} . This function is the effective potential (up to a constant). When the potential is quadratic, it is

$$\mathcal{F}^{\pm}_{\mathrm{H}}(\mathrm{x}) = \int_{2}^{|\mathrm{x}|} \sqrt{\mathrm{t}^2 - 4} \mathrm{d}\mathrm{t}$$
 ,

i.e. if $\lambda_1^{(n)} > \lambda_2^{(n)} > \cdots > \lambda_r^{(n)}$, then

 $\mathbb{P}\left(\lambda_{1}^{(n)} \cong \mathsf{E}_{1}^{+}, \dots, \lambda_{r}^{(n)} \cong \mathsf{E}_{r}^{+}\right) = \exp -n\left[\sum_{1}^{r} \mathcal{F}_{\mathsf{H}}^{+}(\mathsf{E}_{i}^{+}) + o(1)\right]$

Theorem (GNR '16)

Under assumptions on $V,\,(\mu_w^{(n)})$ satisfies the LDP in the scale N with good rate function

$$\mathbb{J}_{\mathsf{meas}}(\mu) = \mathcal{K}(\mu_V \mid \mu) + \sum_k \mathfrak{F}_V(\mathsf{E}_k^+) + \sum_k \mathfrak{F}_V(\mathsf{E}_k^-)$$

for probability measures μ on $\mathbb R$ satisfying

$$\textit{Supp}(\Sigma) = [a_V, b_V] \cup \{E_k^-\}_{k=1}^{K^-} \cup \{E_k^+\}_{j=1}^{K^+}$$

where K^+ (resp. K^-) is 0, finite or infinite and $E_k^- \uparrow a_V$ and $E_k^+ \downarrow b_V$ are isolated points of the support, μ_V is the equilibrium measure and $[a_V, b_V]$ is its support.

We have 2 LDPs for the same object under two different encodings

 I_{coeff} via the coefficients encoding and I_{meas} via the weight + support encoding. By uniqueness we obtain $I_{coeff}(\mu) = I_{meas}(\mu)$. We recover the Killip-Simon sum rule (2003)

$$\mathcal{K}(SC|\mu) + \sum_{k} \mathcal{F}(E_k^+) + \sum_{k} \mathcal{F}(E_k^-) = \sum_{j} G(a_j^2) + \frac{b_j^2}{2},$$

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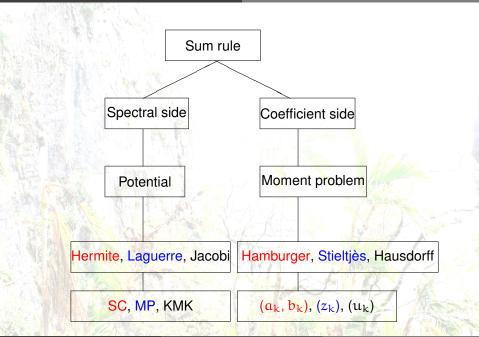
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On the unit circle ${\mathbb T}$ we have three sum rules :

Szegő-Verblunsky formula (1915-1936)

$$\mathcal{K}(\text{UNIF} \mid \mu) = \sum_{j} - \log(1 - |\alpha_{j}|^{2})$$

Gross Witten, with potential $\mathcal{V}(e^{i\theta}) = g \cos \theta$. Hua-Pickreit, with potential $\mathcal{V}(e^{i\theta}) = d \log|1 - e^{i\theta}|^2$.

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Merci pour votre attention! Thanks for your attention!

and

Spectral measures



A. Rouault (LMV)