New estimation of Sobol' indices using kernels

Joint work with Fabrice, Thierry Klein, Clémentine Prieur, and Sébastien da Veiga

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TOULOUSE - FRANCE

Journées MascotNUM, Le Croisic, 4-6 april 2023

Outline of the talk

Framework

In this talk, we consider the following black-box model :

$$Y = f(V_1, \dots, V_p),$$

where $f : E = E_1 \times E_2 \times \cdots \times E_p \to \mathbb{R}^k$ is an unknown and deterministic function.

Main assumptions

1. $V_1, \dots, V_p$ are independent.
2. $\mathbb{E}[\| Y \|^2] < \infty$.
3. Y is scalar (here, for sake of simplicity).

The so-called Sobol' indices

Classically to quantify the amount of randomness that a variable or a group of variables bring to Y, one computes the so-called Sobol' indices.

For instance, the first order Sobol' and the total Sobol' indices with respect to $V_{\mathbf{u}} = (V_i, i \in \mathbf{u})$ is given by

$$S^{\mathbf{u}} = \frac{\mathrm{Var}(\mathbb{E}[Y|V_{\mathbf{u}}])}{\mathrm{Var}(Y)} \quad \textit{and} \quad S^{\mathbf{u},Tot} = 1 - S^{\sim\mathbf{u}} = 1 - \frac{\mathrm{Var}(\mathbb{E}[Y|V_{\sim\mathbf{u}}])}{\mathrm{Var}(Y)}$$

(assuming Y is scalar).

Such indices stem from the Hoeffding decomposition of the variance of f (or equivalently Y) that is assumed to lie in L^2.

Outline of the talk

Pick-Freeze estimation of Sobol' indices (I)

To fix ideas assume for example $p=5$, $\mathbf{u}=\{1,2\}$ so that $\sim \mathbf{u}=\{3,4,5\}$.

We consider the Pick-Freeze variable $Y_{\mathbf{u}}$ defined as follows :

- draw $V=(V_1, V_2, V_3, V_4, V_5)$,
- build $V^{\mathbf{u}}=(V_1, V_2, V_3', V_4', V_5')$.

Then, we compute

- $Y=f(V)$,
- $Y^{\mathbf{u}}=f(V^{\mathbf{u}})$.

A small miracle

$$\mathrm{Var}(\mathbb{E}[Y|X])=\mathrm{Var}(\mathbb{E}[Y|V_{\mathbf{u}}])=\mathrm{Cov}(Y, Y^{\mathbf{u}}) \text{ so that } S^{\mathbf{u}}=\frac{\mathrm{Cov}(Y,Y^{\mathbf{u}})}{\mathrm{Var}(Y)}.$$

Pick-Freeze estimation of Sobol' indices (II)

In practice, generate two n-samples :

- one n-sample of V : $(V_j)_{j=1,...,n}$,
- one n-sample of $V^{\mathbf{u}}$: $\left(V_j^{\mathbf{u}}\right)_{j=1,...,n}$.

Compute the code on both samples :

- $Y_j = f(V_j)$ for $j = 1, \ldots, n$,
- $Y_j^{\mathbf{u}} = f(V_j^{\mathbf{u}})$ for $j = 1, \ldots, n$.

Then estimate $S^{\mathbf{u}}$ by

$$S_{n,PF}^{\mathbf{u}} = \frac{\frac{1}{n}\sum_{j=1}^{n} Y_j Y_j^{\mathbf{u}} - \left(\frac{1}{n}\sum_{j=1}^{n} Y_j\right)\left(\frac{1}{n}\sum_{j=1}^{n} Y_j^{\mathbf{u}}\right)}{\frac{1}{n}\sum_{j=1}^{n}(Y_j)^2 - \left(\frac{1}{n}\sum_{j=1}^{n} Y_j\right)^2}$$

Pick-Freeze scheme (III) : some statistical properties

Is the Pick-Freeze estimator a "good" estimator of the Sobol' index ?

- Is it consistent ? Response : YES SLLN.
- If yes, at which rate of convergence ? Resp. : YES CLT (cv in $\sqrt{n}$).
- Is it asymptotically efficient ? Resp. : YES.
- Is it possible to measure its performance for a fixed n ? Response : YES Berry-Esseen and/or concentration inequalities.

Ref. : A. Janon, T. Klein, A. Lagnoux, M. Nodet, and C. Prieur. " Asymptotic normality et efficiency of a Sobol' index estimator", *ESAIM P&S*, 2013.

F. Gamboa, A. Janon, T. Klein, A. Lagnoux, and C. Prieur. " Statistical Inference for Sobol' Pick Freeze Monte Carlo method", *Statistics*, 2015.

Drawbacks of the Pick-Freeze estimation

- The cost (= number of evaluations of the function f) of the estimation of the p first-order Sobol' indices is quite expensive : $(p+1)n$.
- This methodology is based on a particular design of experiment that may not be available in practice. For instance, when the practitioner only has access to real data.

☞ *We are interested in an estimator based on a n-sample only.*

Mighty estimation based on ranks (I)

Here we assume that the inputs V_i for $i=1,\dots,p$ are scalar and we want to estimate the Sobol' index S^i with respect to $X=V_i$:

$$S^i = \frac{\mathrm{Var}(\mathbb{E}[Y|V_i])}{\mathrm{Var}(Y)} = \frac{\mathrm{Var}(\mathbb{E}[Y|X])}{\mathrm{Var}(Y)}.$$

To do so, we consider a n-sample of the input/output pair (X, Y) given by

$$(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n).$$

The pairs $(X_{(1)}, Y_{(1)}), (X_{(2)}, Y_{(2)}), \dots, (X_{(n)}, Y_{(n)})$ are rearranged in such a way that

$$X_{(1)} < \dots < X_{(n)}.$$

Mighty estimation based on ranks (II)

We introduce

$$S^i_{n,Rank} = \frac{\frac{1}{n}\sum_{j=1}^{n-1} Y_{(j)} Y_{(j+1)} - \left(\frac{1}{n}\sum_{j=1}^{n} Y_j\right)^2}{\frac{1}{n}\sum_{j=1}^{n} Y_j^2 - \left(\frac{1}{n}\sum_{j=1}^{n} Y_j\right)^2}.$$

Statistical properties

- Consistency : OK.
- Central Limit Theorem : OK.

Ref. : S. Chatterjee. "A new coefficient of Correlation", *JASA*, 2020.
F. Gamboa, P. Gremaud, T. Klein, and A. Lagnoux. " Global Sensitivity Analysis : a new generation of mighty estimators based on rank statistics", *Bernoulli.* 2022.

Efficient estimation based on kernels

Ref. : S. da Veiga and F. Gamboa. “Efficient estimation of sensitivity indices”, *Journal of Nonparametric Statistics*, 2013.

Here again we assume that the inputs V_i for $i = 1, \ldots, p$ are scalar.

To do so, the initial n-sample is split into two samples of sizes

- $n_1 = \lfloor n / \log n \rfloor \Rightarrow$ estimation of the joint density of (V, Y)
- $n_2 = n - n_1 \approx n \Rightarrow$ Monte-Carlo estimation of the integral involved in the quantity of interest.

Statistical properties

- Consistency : OK.
- Central Limit Theorem : OK.
- Asymptotic efficiency : OK.

Estimation based on nearest neighbors

Ref. : L. Devroye, L. Györfi, G. Lugosi, and H. Walk. “A nearest neighbor estimate of the residual variance”, *EJS*, 2018.

Here the input X with respect we want to compute the Sobol' index is allowed to have dimension d.

To do so, the initial n-sample is split into two samples of sizes

- $n/2 \Rightarrow$ estimation of the regression function $\mathbb{E}[Y|X=x]$ using the first NN of x among the points of the first sample ;
- $n/2 \Rightarrow$ plug-in estimator.

Statistical properties

- Consistency : OK
- Central Limit Theorem : OK only for $d \leqslant 3$.

Outline of the talk

Introduction

Recall that

$$S^X = \frac{\mathrm{Var}(\mathbb{E}[Y|X])}{\mathrm{Var}(Y)}$$

allowing a multidimensional X living in a compact set : $X \in \mathscr{D} \subset \mathbb{R}^d$.

To estimate $\mathbb{E}[Y]$ and $\mathrm{Var}(Y)$ from the n-sample $(Y_j)_{j=1,\dots,n}$ of the output Y, we will naturally use the classical empirical mean and variance respectively.

☞ Thus we focus on the estimation of $\mathbb{E}[\mathbb{E}[Y|X]^2]$ from the n-sample $(X_j, Y_j)_{j=1,\dots,n}$ of the pair (X, Y).

Introduction

A natural estimator inspired from the NN and kernel-based plug-in estimators would be

$$\binom{n}{2}^{-1} \sum_{1 \leqslant j < j' \leqslant n} \frac{Y_j Y_{j'}}{2} \left(\frac{K_{h_n}(X_{j'} - X_j)}{f_X(X_j)} + \frac{K_{h_n}(X_j - X_{j'})}{f_X(X_{j'})} \right)$$

for a bandwidth $h_n > 0$ and a kernel K_{h_n}.

Nevertheless, boundary issues appear when the input domain is compact.

New estimation based on kernels

To bypass this issue, we consider the following kernel-based estimator

$$T_{n,h_n} = \binom{n}{2}^{-1} \sum_{1 \leqslant j < j' \leqslant n} \frac{Y_j Y_{j'}}{2} \left(\frac{K_{h_n} \circ A_{X_j}(X_{j'} - X_j)}{f_X(X_j)} + \frac{K_{h_n} \circ A_{X_{j'}}(X_j - X_{j'})}{f_X(X_{j'})} \right).$$

for a bandwidth $h_n > 0$, a mirror-type transformation A_x, and a kernel K_{h_n}.

We introduce the functions

$$g_1(x) = \mathbb{E}[Y|X = x] \quad \text{and} \quad g_2(x) = \mathbb{E}[Y^2|X = x].$$

The supremum norm is denoted by $\|\cdot\|_\infty$.

Multi-index notation and smoothness

For any d and $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{R}_+^d$, we define the integer part of β by

$$\lfloor \beta \rfloor = \left(\lfloor \beta_1 \rfloor, \dots, \lfloor \beta_d \rfloor \right) =: \gamma \in \mathbb{N}^d.$$

In addition, we introduce, for any $v \in \mathbb{R}^d$,

$$|\gamma| = \gamma_1 + \cdots + \gamma_d, \quad \gamma! = \gamma_1! \dots \gamma_d!, \quad \text{and} \quad v^\beta = v_1^{\beta_1} \dots v_d^{\beta_d}.$$

Let $\alpha > 0$. We define $\mathscr{C}^\alpha(\mathscr{D}) = \{\phi : \mathscr{D} \to \mathbb{R}$ with derivatives up to order $\lfloor \alpha \rfloor$ and partial derivative of order $\lfloor \alpha \rfloor$ is $\alpha - \lfloor \alpha \rfloor$-Hölder$\}$. Namely, there exists $C_\phi > 0$ such that, for any x and $x' \in \mathscr{D}$, one has

$$\left| \frac{\partial^\beta \phi}{\partial x^\beta}(x) - \frac{\partial^\beta \phi}{\partial x^\beta}(x') \right| \leqslant C_\phi \left\| x - x' \right\|_\infty^{\alpha - \lfloor \alpha \rfloor}$$

for any $\beta \in \mathbb{N}^d$ such that $|\beta| = \lfloor \alpha \rfloor$.

Assumptions

($\mathscr{A}1$) Support The support of inputs $V = (V_1, \dots, V_p)$ is of the form $[B_1, C_1] \times \cdots \times [B_p, C_p]$ where $B_i < C_i$ for all $1 \leqslant i \leqslant p$.

($\mathscr{A}2$) Absolute continuity The distribution of the random vector $(V, Y) \in \mathbb{R}^p \times \mathbb{R}$ is absolutely continuous with respect to the Lebesgue measure. The marginal pdf of (X, Y), V, X, and W are denoted by $f_{X,Y}$, f_V, f_X, and f_W respectively.

($\mathscr{A}3$) Kernel Let $K : \mathbb{R}^d \to \mathbb{R}$ be a kernel with support included in $\mathscr{D}$ such that $\|K\|_\infty < +\infty$ and $\int_{\mathscr{D}} K(u)du = 1$. We assume that K is of order $\lfloor \alpha \rfloor$ which means that $\int_{\mathscr{D}} u^\beta K(u)du = 0$ for any $\beta \in \mathbb{N}^d$ such that $0 < |\beta| \leqslant \lfloor \alpha \rfloor$. Finally, we define $K_h(x) = K(x/h)/h^d$ for any $x \in \mathscr{D}$.

($\mathscr{A}4$) Bandwidth The sequence $(h_n)_{n \in \mathbb{N}}$ of bandwidths is positive and $h_n \to 0$ as $n \to \infty$.

Mirror-type transformation

The next definition allows to circumvent the boundary issues.

($\mathscr{D}$1) For $x \in \mathscr{D}$, we define

$$A_x : \begin{cases} \mathbb{R}^d & \longrightarrow & \mathbb{R}^d \\ u = (u_1, \ldots, u_d) & \longmapsto & (\sigma_1(x_1)u_1, \ldots, \sigma_d(x_d)u_d) \end{cases}$$

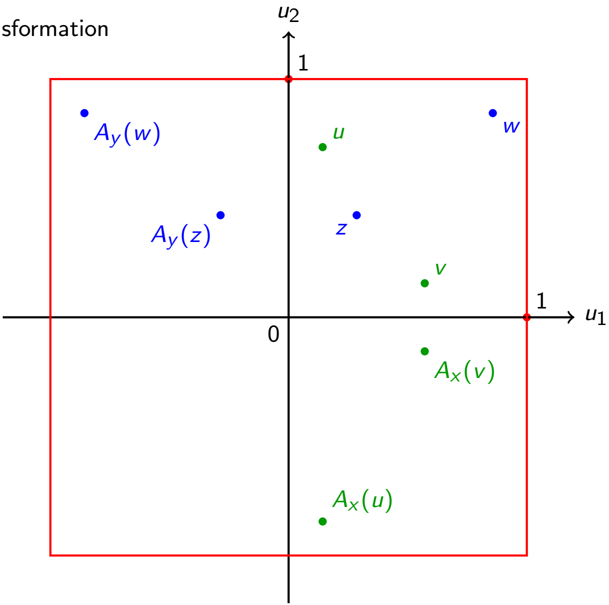
with $\sigma_i(s) := 1 - 2\mathbb{1}_{\left(\frac{B_i + C_i}{2}, C_i\right)}(s) \in \{-1, 1\}$.

Observe that $\mathscr{A} = \{A_x, x \in \mathscr{D}\}$ is a finite subset of $GL_d(\mathbb{R})$, $\mathscr{A} = \{A_1, \ldots, A_\kappa\}$, with cardinal $\kappa = 2^d$. Moreover, it satisfies

- (i) for any $\ell = 1, \ldots, \kappa$, $\left|\det(A_\ell)\right| = 1$;
- (ii) Mirror condition : for any $x \in \mathscr{D}$, there exists $A_\ell \in \mathscr{A}$ such that $A_x = A_\ell$ and $x + A_x^{-1}([0, 1/2]^d) \subset \mathscr{D}$.

Mirror-type transformation

$\mathscr{D} = [0,1]^2$
$x = (1/3, 3/4)$
$y = (2/3, 1/5)$



Theorem (Bias and quadratic controls)

I.1. Assume that $g_1 \in \mathbb{L}^1(\mathscr{D})$, $g_1 f_X \in \mathscr{C}^\alpha(\mathscr{D})$, *and* $\int_{\mathscr{D}} \left| g_1(x) \frac{\partial^\beta (g_1 f_X)}{\partial x^\beta}(x) \right| dx < \infty$ *for any* β *such that* $1 \leqslant |\beta| < \lfloor \alpha \rfloor$. *Then we have*

$$\left| \mathbb{E}[T_{n,h_n}] - \mathbb{E}[\mathbb{E}[Y|X]^2] \right| \leqslant C h_n^\alpha.$$

I.2. Assume in addition that $g_1 \in \mathscr{C}^\alpha(\mathscr{D})$, $g_2 / f_X \in \mathbb{L}^1(\mathscr{D}) \cap \mathbb{L}^2(\mathscr{D})$ *and* $g_2 f_X \in \mathbb{L}^2(\mathscr{D})$, *and* $\int_{\mathscr{D}} g_2(x) \left| \frac{\partial^\beta (g_1 f_X)}{\partial x^\beta}(x) \frac{\partial^{\beta'} (g_1 f_X)}{\partial x^{\beta'}}(x) \right| dx < \infty$ *for any* β *and* β' *such that* $1 \leqslant |\beta|, |\beta'| < \lfloor \alpha \rfloor$. *Then we have*

$$\mathbb{E}\left[\left(T_{n,h_n} - \mathbb{E}[T_{n,h_n}] - \frac{1}{n} \sum_{j=1}^n Z_j \right)^2 \right] \leqslant C h_n^{2\alpha} + \frac{C}{h_n^d n^2}.$$

where, for $j = 1, \ldots, n$, $Z_j = 2(Y_j g_1(X_j) - \mathbb{E}[\mathbb{E}[Y|X]^2])$.

Theorem (Central Limit Theorem)

II. Assuming in addition that $\mathbb{E}[Y^4] < \infty$, $\alpha > d/2$, $h_n \underset{n\to\infty}{\to} 0$, $nh_n^d \underset{n\to\infty}{\to} \infty$, *and* $nh_n^{2\alpha} \underset{n\to\infty}{\to} 0$, *we get*

$$\sqrt{n}\Big(T_{n,h_n} - \mathbb{E}[\mathbb{E}[Y|X]^2]\Big) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, 4\tau^2)$$

with $\tau^2 = \mathrm{Var}(Yg_1(X))$.

Ref. : F. Gamboa, T. Klein, A. Lagnoux, C. Prieur, and S. da Veiga. "New estimation of Sobol' indices based on kernels". Available on Hal and Arxiv (2023). https://hal.science/hal-04052837.

Using the delta method, we are now able to get the asymptotic behavior of the estimation of S^X.

Corollary (CLT for the estimation of the Sobol' indices)

Under all the assumptions of the theorem (II included), one has

$$\sqrt{n}\left(\frac{T_{n,h_n}-\left(\frac{1}{n}\sum_{j=1}^{n}Y_j\right)^2}{\frac{1}{n}\sum_{j=1}^{n}Y_j^2-\left(\frac{1}{n}\sum_{j=1}^{n}Y_j\right)^2}-S^X\right)\xrightarrow[n\to\infty]{\mathscr{D}}\mathcal{N}(0,\sigma^2),$$

where the limit variance σ^2 has an explicit expression.

Using one more time the delta method, we deduce the asymptotic behavior of the vector containing the p first-order Sobol' indices. Let us denote S^i the first-order Sobol index associated to the i-th input and its estimator $\widehat{S}^i$ given by :

$$\widehat{S}^i = \frac{T_{n,h_n} - \left(\frac{1}{n}\sum_{j=1}^n Y_j\right)^2}{\frac{1}{n}\sum_{j=1}^n Y_j^2 - \left(\frac{1}{n}\sum_{j=1}^n Y_j\right)^2}.$$

Corollary (CLT for the global estimation of the p first-order Sobol' indices)

Under all the assumptions of the theorem (II included), one has

$$\sqrt{n}\Big((\widehat{S}^1,\dots,\widehat{S}^p)^T - (S^1,\dots,S^p)^T\Big) \xrightarrow[n\to\infty]{\mathscr{D}} \mathcal{N}(0,\Sigma),$$

where the limit variance Σ has an explicit expression.

Also in the paper...

- A procedure for bandwidth selection inspired from Delyon and Portier in 2016.
- An extension to unknown density f_X in which we consider
  - parametric estimation of f_X,
  - nonparametric estimation of f_X.

Outline of the talk

Sketch of the proof (I)

Since $Y \in \mathbb{L}^2(\mathbb{R})$, one has

$$\begin{aligned}\mathbb{E}[T_{n,h}] &= \iint_{\mathscr{D}^2} \frac{K_h \circ A_{x_1}(x_2 - x_1)}{f_X(x_1)} f_X(x_1) f_X(x_2) g_1(x_1) g_1(x_2) dx_1 dx_2 \\ &= \int_{\mathscr{D}} \int_{\mathscr{D}_x} K(u) g_1(x) g_1(x + hA_x^{-1}(u)) f_X(x + hA_x^{-1}(u)) du dx.\end{aligned}$$

In addition,

$$\mathbb{E}[\mathbb{E}[Y|X]^2] = \mathbb{E}[g_1^2(X)] = \int_{\mathscr{D}} g_1(x)^2 f_X(x) dx = \iint_{\mathscr{D}^2} K(u) g_1(x)^2 f_X(x) dx du$$

leading to $\mathbb{E}[T_{n,h}] - \mathbb{E}[\mathbb{E}[Y|X]^2]$

$$= \iint_{\mathscr{D}^2} K(u) g_1(x) \Big(g_1(x + hA_x^{-1}(u)) f_X(x + hA_x^{-1}(u)) - g_1(x) f_X(x) \Big) dx du.$$

Sketch of the proof (II)

For all $j, j' = 1, \ldots, n$, we introduce the symmetric function given by

$$R\left(\begin{pmatrix} x_j \\ y_j \end{pmatrix}, \begin{pmatrix} x_{j'} \\ y_{j'} \end{pmatrix}\right) = \frac{y_j y_{j'}}{2}\left(\frac{K_{h_n} \circ A_{x_j}(x_{j'} - x_j)}{f_X(x_j)} + \frac{K_{h_n} \circ A_{x_{j'}}(x_j - x_{j'})}{f_X(x_{j'})}\right).$$

Then the Hoeffding projections of R are given by (see Pena 1999)

$$\pi_1 R \begin{pmatrix} x \\ y \end{pmatrix} = \mathbb{E}\left[R\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}\right)\right] - \mathbb{E}\left[R\left(\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}\right)\right]$$

$$\pi_2 R\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = R\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) - \mathbb{E}\left[R\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}\right)\right] - \mathbb{E}\left[R\left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}\right)\right] + \mathbb{E}\left[R\left(\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}\right)\right].$$

Sketch of the proof (III)

Hence the Hoeffding decomposition writes

$$
\begin{aligned}
T_{n,h_n} - \mathbb{E}[T_{n,h_n}] &= 2U_n^{(1)}(\pi_1 R) + U_n^{(2)}(\pi_2 R) \\
&= \frac{1}{n}\sum_{j=1}^{n} \underbrace{2(Y_j g_1(X_j) - \mathbb{E}[\mathbb{E}[Y|X]^2])}_{=Z_j} \\
&\quad + \underbrace{2U_n^{(1)}(\pi_1 R) - \frac{2}{n}\sum_{j=1}^{n}(Y_j g_1(X_j) - \mathbb{E}[\mathbb{E}[Y|X]^2])}_{=S_1} + \underbrace{U_n^{(2)}(\pi_2 R)}_{=S_2}.
\end{aligned}
$$

with $U_n^{(1)}(\pi_1 R) = \frac{1}{n}\sum_{j=1}^{n}(\pi_1 R)\begin{pmatrix} X_j \\ Y_j \end{pmatrix}$

$$U_n^{(2)}(\pi_2 R) = \frac{2}{n(n-1)} \sum_{1 \leqslant j < j' \leqslant n} (\pi_2 R)\left(\begin{pmatrix} X_j \\ Y_j \end{pmatrix}, \begin{pmatrix} X_{j'} \\ Y_{j'} \end{pmatrix}\right)$$ (Giné, Nikl '08).

Outline of the talk

Ishigami function

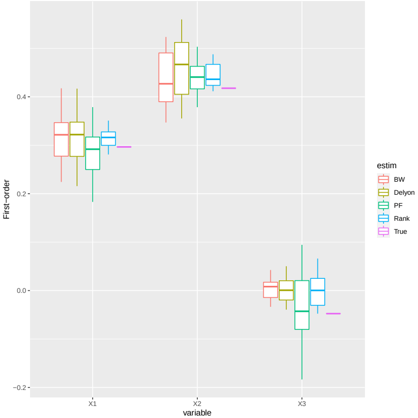
The Ishigami model is given by :

$$Y = f(V) = f(V_1, V_2, V_3) = \sin(V_1) + 7\sin^2(V_2) + 0.1 V_3^4 \sin(V_1)$$

where $(V_j)_{j=1,2,3}$ are i.i.d. uniform random variables on $[-\pi;\pi]$.

One has

$$S^1 = 0.3139,\ \ S^2 = 0.4424,\ \ S^3 = 0.$$


First−order
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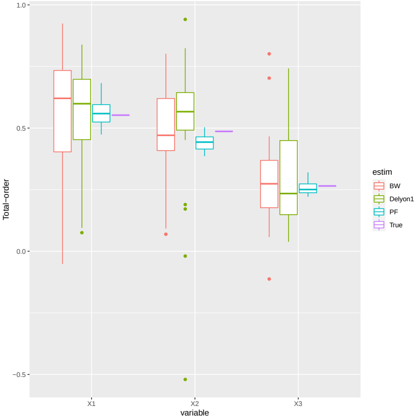
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Bratley function

Let us consider the Bratley function defined by :

$$g(V_1,\dots,V_p)=\sum_{i=1}^{p}(-1)^i\prod_{j=1}^{i}V_j,$$

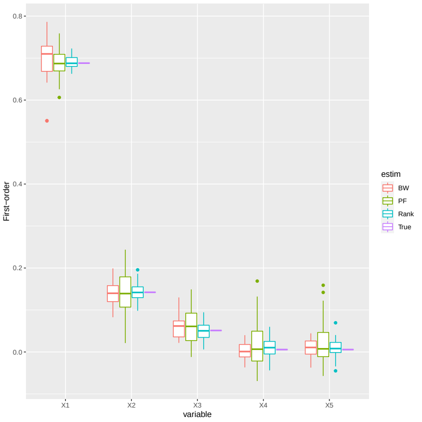
with $V_i \sim \mathscr{U}([0,1])$ i.i.d. After some tedious calculations, one gets

$$\mathrm{Var}(Y)=\frac{1}{18}-\frac{2}{45}\left(-\frac{1}{2}\right)^p+\frac{1}{10}\frac{1}{3^p}-\frac{1}{9}\frac{1}{2^{2p}}$$

$$S^i=\frac{\mathrm{Var}[\mathbb{E}(Y|V_i)]}{\mathrm{Var}(Y)}=\frac{1}{\mathrm{Var}(Y)}\frac{\left(2^{p-i+1}-(-1)^{p-i+1}\right)^2}{2^{2p}\times 3^3}.$$

Now let us compute the total indices for $i=1$ and 2 and $p=5$,

$$S^{1,\mathrm{Tot}}=1-\frac{1111}{3^4\times 2^{10}\times \mathrm{Var}(Y)}\approx 0.77,\quad S^{2,\mathrm{tot}}=1-\frac{3703}{3^4\times 2^{10}\times \mathrm{Var}(Y)}\approx 0.22.$$


First−order
0.8
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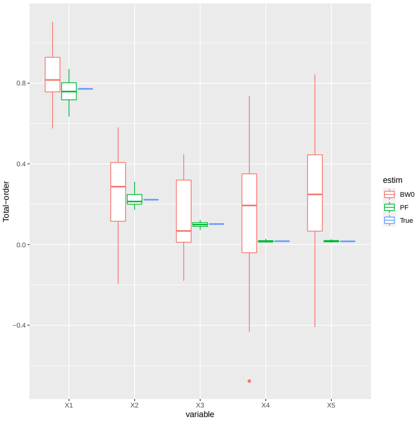
Total−order
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