A giant python and a lazy elephant talking in a Chinese restaurant

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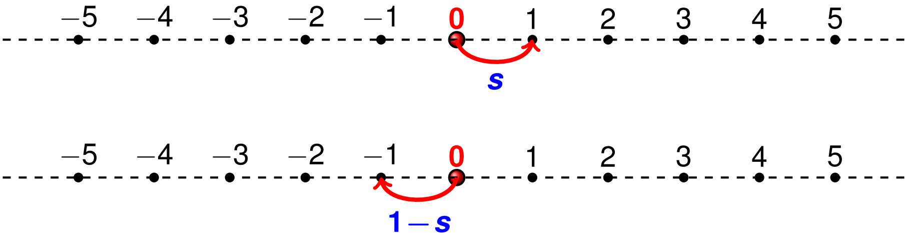
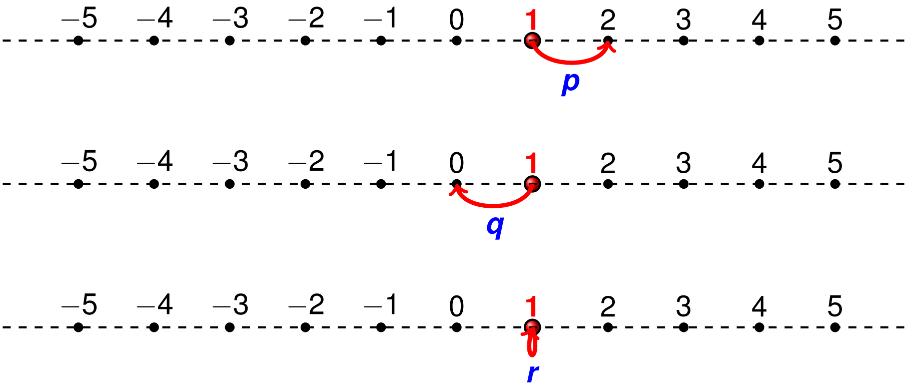
Plan

Outline

Elephants always remember where they have been

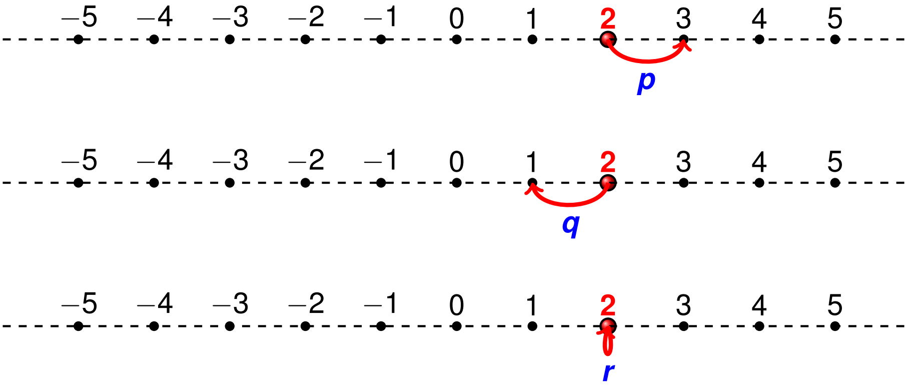


Elephants always remember where they have been



$$p + q + r = 1$$

Elephants always remember where they have been



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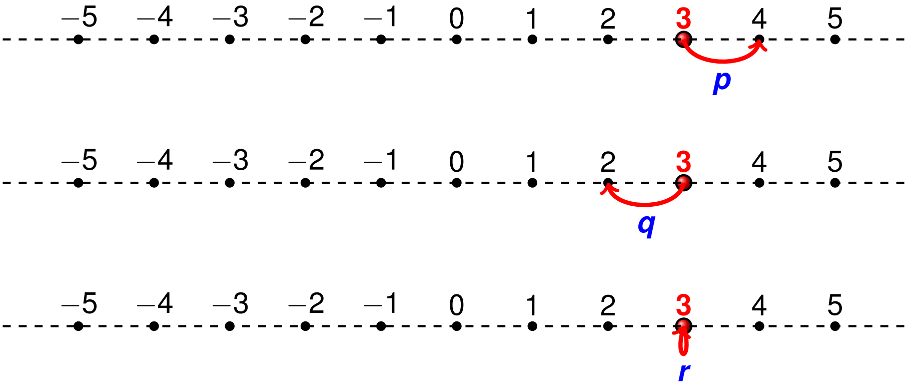

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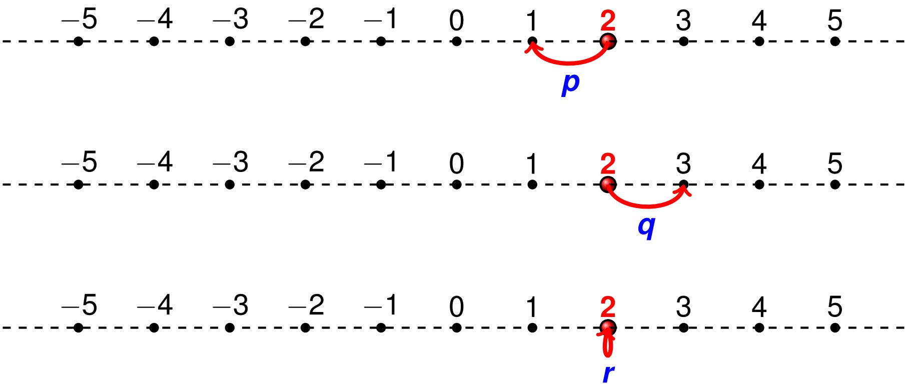


Steps of the lazy elephant

Steps of the lazy elephant

For all $n \geq 1$,

$$X_{n+1} = \begin{cases} +X_k & \text{with probability} \quad p, \\ -X_k & \text{with probability} \quad q, \\ 0 & \text{with probability} \quad r, \end{cases}$$

where the integer k is chosen uniformly at random among $\{1, \ldots, n\}$.

Positions of the lazy elephant random walk

The position of the lazy elephant at time $n \geq 0$ is given by

$$S_{n+1} = S_n + X_{n+1}.$$

Another quantity of interest is given by the number of ones

$$\Sigma_{n+1} = \Sigma_n + X_{n+1}^2.$$

We have almost surely for all $n \geq 1$,

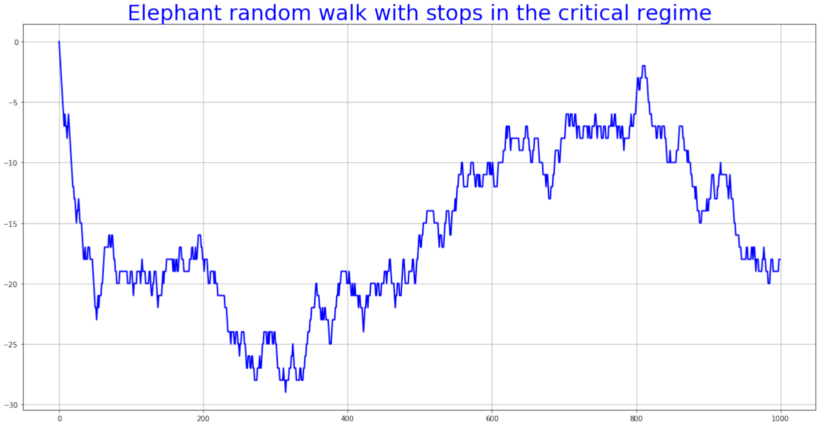
$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = a\frac{S_n}{n} \qquad \text{and} \qquad \mathbb{E}[X_{n+1}^2|\mathcal{F}_n] = b\frac{\Sigma_n}{n}$$

where $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$,

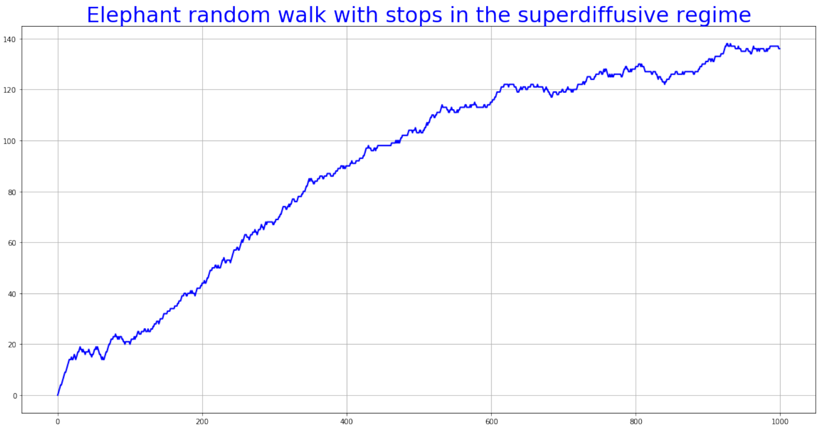
$$a = p - q, \qquad b = p + q.$$

ERWS in the diffusive regime $p < 3(1-r)/4$.

ERWS in the critical regime $p = 3(1 - r)/4$.



ERWS in the superdiffusive regime $p > 3(1-r)/4$.



Our martingale approach

We have almost surely for all $n \geq 1$,

$$\begin{aligned} \mathbb{E}[S_{n+1}|\mathcal{F}_n] &= \mathbb{E}[S_n + X_{n+1}|\mathcal{F}_n] = \alpha_n S_n, \\ \mathbb{E}[\Sigma_{n+1}|\mathcal{F}_n] &= \mathbb{E}[\Sigma_n + X_{n+1}^2|\mathcal{F}_n] = \beta_n \Sigma_n, \end{aligned}$$

where

$$\alpha_n = 1 + \frac{a}{n} \qquad \text{and} \qquad \beta_n = 1 + \frac{b}{n}.$$

Let (a_n) and (b_n) be the sequences given for $n \geq 2$ by

$$a_n = \prod_{k=1}^{n-1} \alpha_k^{-1} = \frac{\Gamma(n)\Gamma(a+1)}{\Gamma(n+a)}, \qquad b_n = \prod_{k=1}^{n-1} \beta_k^{-1} = \frac{\Gamma(n)\Gamma(b+1)}{\Gamma(n+b)}$$

where Γ stands for the Euler Gamma function. Denote

$$M_n = a_n S_n \qquad \text{and} \qquad N_n = b_n \Sigma_n.$$

Our martingale approach

Since

$$a_{n+1} = \prod_{k=1}^{n} \alpha_k^{-1} = \alpha_n^{-1} a_n,$$

we have almost surely

$$\begin{aligned}
\mathbb{E}[M_{n+1}|\mathcal{F}_n] &= \mathbb{E}[a_{n+1}S_{n+1}|\mathcal{F}_n], \\
&= a_{n+1}\mathbb{E}[S_{n+1}|\mathcal{F}_n], \\
&= a_{n+1}\alpha_n S_n, \\
&= M_n.
\end{aligned}$$

It means that (M_n) as well as (N_n) are two **martingale** sequences.

A critical value

It is well-known that

$$\lim_{n\to\infty} \frac{\Gamma(n+a)}{\Gamma(n)n^a} = 1$$

which implies that

$$\lim_{n\to\infty} n^a a_n = \Gamma(a+1).$$

Moreover,

$$a < \frac{b}{2} \Longleftrightarrow p_r < 3/4, \quad a = \frac{b}{2} \Longleftrightarrow p_r = 3/4, \quad a > \frac{b}{2} \Longleftrightarrow p_r > 3/4$$

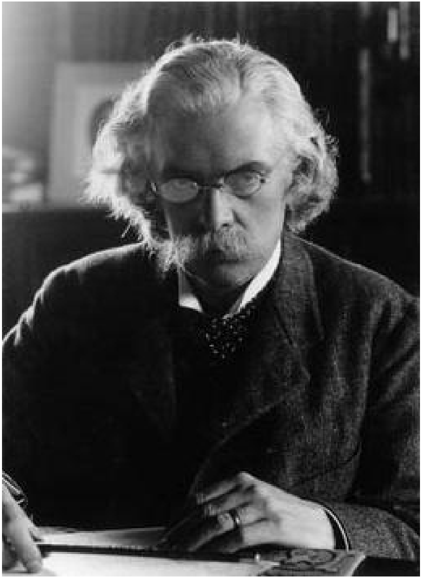
where the critical value

$$\boldsymbol{p_r = \frac{p}{1-r}}.$$

Definition

The ERW with stops is **diffusive** if $p_r < 3/4$, **critical** if $p_r = 3/4$, **superdiffusive** if $p_r > 3/4$.

Outline



The Mittag-Leffler function was introduced at the beginning of the last century. It is defined, for all $z \in \mathbb{C}$, by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+n\alpha)}$$

where α is a positive real parameter and Γ stands for the Euler Gamma function. One can observe that $E_1(z) = \exp(z)$ while $E_2(-z^2) = \cos(z)$.

Definition

A positive random variable X has a Mittag-Leffler distribution with parameter $\alpha \in [0,1]$ if its Laplace transform is given, for all $t \in \mathbb{R}$, by

$$\mathbb{E}[\exp(tX)] = E_\alpha(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(1+n\alpha)}.$$

Consequently, for any integer $m \geq 1$,

$$\mathbb{E}[X^m] = \frac{m!}{\Gamma(1+m\alpha)}.$$

The Mittag-Leffler distribution satisfies the **Carleman's condition** which means that it is characterized by its moments. If $X \sim \mathcal{ML}(\alpha)$ with $0 < \alpha < 1$, its probability density function is given by

$$f_\alpha(x) = \frac{1}{\pi\alpha}\sum_{n=0}^{\infty} \Gamma(1+\alpha n)\sin(\alpha n\pi)\frac{(-x)^{n-1}}{n!}\mathrm{I}_{\{x>0\}}.$$

The only simplification occurs for $\alpha = 1/2$,

$$f_{1/2}(x) = \frac{1}{\sqrt{\pi}}\exp\Big(-\frac{x^2}{4}\Big)\mathrm{I}_{\{x>0\}}.$$

It means that $\mathcal{ML}(1/2)$ coincides with the distribution of $|Y|$ where Y has a Gaussian $\mathcal{N}(0,2)$ distribution.

A keystone lemma

Lemma

Whatever the values of the parameters p, q in $[0,1]$ *and r in* $]0,1[$, *we have the almost sure convergence*

$$\lim_{n\to\infty} \frac{1}{n^{1-r}} \Sigma_n = \Sigma \qquad a.s.$$

where $\Sigma \sim \mathcal{ML}(1-r)$. *Consequently,* Σ *is positive with probability one. Moreover, this convergence holds in* $\mathbb{L}^m$ *for any integer* $m \geq 1$,

$$\mathbb{E}[\Sigma^m] = \frac{m!}{\Gamma(1+m(1-r))}.$$

Outline

Strong law of large numbers

We focus our attention on the diffusive regime where

$$p_r < \frac{3}{4}.$$

Theorem

We have the almost sure convergence

$$\lim_{n\to\infty} \frac{1}{n} S_n = 0 \qquad a.s.$$

Asymptotic normality

Denote by σ_r^2 the asymptotic variance

$$\sigma_r^2 = \frac{1-r}{3(1-r)-4p}.$$

Theorem

We have the asymptotic normality

$$\frac{S_n}{\sqrt{\Sigma_n}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma_r^2).$$

Moreover, we also have

$$\frac{S_n}{\sqrt{n^{1-r}}} \xrightarrow{\mathcal{L}} \sqrt{\Sigma'}\mathcal{N}(0, \sigma_r^2)$$

where Σ' is independent of the Gaussian $\mathcal{N}(0, \sigma_r^2)$ random variable and Σ' has a Mittag-Leffler distribution with parameter $1-r$.

Strong law of large numbers

Let's talk now on the critical regime where

$$p_r = \frac{3}{4}.$$

Theorem

We have the almost sure convergence

$$\lim_{n\to\infty} \frac{1}{n} S_n = 0 \qquad a.s.$$

Asymptotic normality

Theorem

We have the asymptotic normality

$$\frac{S_n}{\sqrt{\Sigma_n \log \Sigma_n}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1).$$

Moreover, we also have

$$\frac{S_n}{\sqrt{n^{1-r}\log n}} \xrightarrow{\mathcal{L}} \sqrt{(1-r)\Sigma'}\mathcal{N}(0,1)$$

where Σ' *is independent of the Gaussian* $\mathcal{N}(0,1)$ *random variable and* Σ' *has a Mittag-Leffler distribution with parameter* $1-r$.

Almost sure convergence

We focus on the more attractive superdiffusive regime where

$$p_r > \frac{3}{4}.$$

Theorem

We have the almost sure convergence

$$\lim_{n\to\infty} \frac{1}{n^{2p+r-1}} S_n = L \qquad a.s.$$

where L is a non-degenerate random variable. Moreover, this convergence holds in $\mathbb{L}^m$ *for any integer* $m \geq 1$,

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\frac{S_n}{n^{2p+r-1}} - L\right|^m\right] = 0.$$

Corollary

The first four moments of L are given by

$$\begin{aligned}
\mathbb{E}[L] &= \frac{2s-1}{(2p+r-1)\Gamma(2p+r-1)},\\
\mathbb{E}[L^2] &= \frac{1}{(4p+3(r-1))\Gamma(2(2p+r-1))},\\
\mathbb{E}[L^3] &= \frac{2p(2s-1)}{(2p+r-1)(4p+3(r-1))\Gamma(3(2p+r-1))},\\
\mathbb{E}[L^4] &= \frac{6\big(8p^2-4p(1-r)-(1-r)^2\big)}{(8p+5(r-1))(4p+3(r-1))^2\Gamma(4(2p+r-1))}.
\end{aligned}$$

Outline

Pitman's Chinese restaurant

At stage n, k tables are occupied in the Chinese restaurant, one with a giant python and a lazy elephant ☺

Table 1 n_1 — $\frac{n_1 - b}{n+\theta}$

Table 2 n_2 — $\frac{n_2 - b}{n+\theta}$

Table 3 n_3 — $\frac{n_3 - b}{n+\theta}$

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Table k n_k — $\frac{n_k - b}{n+\theta}$

New Table — $\frac{\theta + bk}{n+\theta}$

Pitman's Chinese restaurant

1. The next customer goes to an occupied table i with probability
$$\frac{n_i - b}{n + \theta}.$$
2. The next customer goes to a new table with probability
$$\frac{\theta + bk}{n + \theta}.$$

Assume that $b + \theta > 0$ and $0 < b < 1$. Denote by Σ_n the number of tables in the Chinese restaurant.

Theorem (Pitman, PTRF 1995)

We have the almost sure convergence

$$\lim_{n\to\infty} \frac{1}{n^b}\Sigma_n = \Sigma \qquad a.s.$$

where the distribution of Σ is closely related with the $\mathcal{ML}(b)$.

!!!! Happy Birthday Fabrizio !!!!